ON THE PROBLEM OF THE GAME-INTERCEPTION OF MOTIONS

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A. I. SUBBOTIN
(Sverdlovsk)

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1. Statement of the problem. We shall consider a problem of interception of completely controlled motions described by the following systems of ordinary differential equations

$$dy / d\tau = Ay + Bu \tag{1.1}$$

$$dz / d\tau = Az + Bv \tag{1.2}$$

Here $y(\tau) = \{y_1(\tau), \ldots, y_n(\tau)\}$ is the phase coordinate vector of the first pursuing object, $z(\tau) = \{z_1(\tau), \ldots, z_n(\tau)\}$ is the phase coordinate vector of the second pursued object; A and B are constant matrices of appropriate dimensions, while u and v are r-dimensional controls constrained [2] by

$$\int_\tau^\infty \|u(t)\|^2 dt \leqslant \mu^2(\tau), \qquad \int_\tau^\infty \|v(t)\|^2 dt \leqslant \nu^2(\tau) \qquad (\tau \geqslant t_0) \tag{1.3}$$

Here and in the following

$$\|w\| = (w_1^2 + \ldots + w_r^2)^{1/2} \tag{1.4}$$

Let at the initial instant $\tau = t_0$

$$y(t_0) = y^\circ,\ z(t_0) = z^\circ,\ y^\circ \neq z^\circ, \quad \mu(t_0) = \mu^\circ > \nu(t_0) = \nu^\circ \tag{1.5}$$

be given. We shall assume that an interception of motions $y(\tau)$ and $z(\tau)$ occurs at the time $\tau = \vartheta$ if at this instant

$$y_i(\vartheta) = z_i(\vartheta) \qquad (i = 1, \ldots, n) \tag{1.6}$$

takes place for the first time.

We shall call the quantity ϑ the instant of interception and $(\vartheta - t_0) = T(t_0)$ the time-to-interception. We consider the interception problem as a game with two players [1] where the criterion used to evaluate the players' actions is the time-to-interception T_{uv}, which depends on the choice of the strategies $u[y, z, \mu, \nu]$ and $v[y, z, \mu, \nu]$. The first player (the pursuer) in choosing his strategy $u[y, z, \mu, \nu]$ restricted by the first of conditions (1.3) seeks to minimize the time-to-interception, while the second player (the pursued) chooses strategies $v[y, z, \mu, \nu]$ satisfying the second of conditions (1.3) such that the motions $y(\tau)$ and $z(\tau)$ do not meet at all or such that the time-to-interception is maximal. In accordance with the game formulation of the problem, optimal behavior of the first player will consist of choosing a strategy $u[y, z, \mu, \nu]$, securing $\min_u \max_v T_{uv}$, while for the second player it will be the choice of a strategy $v[y, z, \mu, \nu]$, yielding $\max_v \min_u T_{uv}$.

It was shown in [2] that in the case of an interception in all coordinates (1.6) under the constraints given by (1.3), the only pursuer strategy securing $\min_u \max_v T_{uv}$, will be the rule of extremal aiming, i.e. aiming, at each instant τ, at the point of contact $c(\tau)$ of regions of attainability of motions $y(\tau)$ and $z(\tau)$, corresponding to the instant of absorption $t = \vartheta_0$. (A control aiming the motion of (1.1) or (1.2) at the point $c(\tau)$ shall, in the following, be denoted by u° or v°, respectively). Thus, $T_{u^\circ v^\circ} \geqslant T_{u^\circ v}$, and equality occurs only when $v = v^\circ$. The same paper [2] contains an example in which the author constructs a control $u = u^*$ such that if the pursued player chooses $v = v^\circ$ and the pursuer $u = u^*$, interception occurs in less than the time $T_{u^\circ v^\circ}$. In exactly the same way we can show that if the pursued player chooses the program control $v = v_{t^\circ}{}^\circ(\tau)$, which at the initial instant $\tau = t_0$ directs the motion of system (1.2) to the point $c(t_0)$, then there exists a control $u = u^*$ such that the motions meet sooner than in the time $T_{u^\circ v^\circ}$. Hence, the inequality $T_{uv^\circ} \geqslant T_{u^\circ v^\circ}$ is generally invalid, and the strategy $v = v^\circ$ does not yield $\max_v \min_u T_{uv}$; moreover, the pair of strategies $u = u^\circ$, $v = v^\circ$ does not yield the saddle point of the game under consideration.

Further on we shall construct the control $v(\tau) = v[y(\tau), z(\tau), \mu(\tau), \nu(\tau), t_0]$ whose choice guarantees that interception will occur not sooner than in the time $T_\varepsilon(t_0) = T_{u^\circ v^\circ}(t_0) - \delta$, (where δ is arbitrarily small) for any permissible behavior of the pursuer. Thus it is proved that

$$\sup_v \inf_u T_{uv}(t_0) = \min_u \max_v T_{uv}(t_0) \tag{1.7}$$

The control $v = v[y(\tau), z(\tau), \mu(\tau), \nu(\tau), t_0]$ is formed at each instant τ from $y(\tau)$, $z(\tau)$, $\mu(\tau)$ and $\nu(\tau)$ existing, and is not influenced by any information on the choice of $u(\tau)$ at that instant. It should be noted that when the state of the system at the time $t = t_0 < \tau$ is taken into account, then an aftereffect element enters the control.

2. Construction of a control. We shall use a method proposed in [3] to construct a required strategy.

We shall compare the problem of interception with the following problem on time-optimal operation: to find, at fixed τ, a control $w(\tau)$ constrained by

$$\int_\tau^\infty \|w(t)\|^2 dt \leqslant \zeta^2(\tau) \tag{2.1}$$

which transfers the system

$$dx/dt = Ax + Bw \tag{2.2}$$

from the position $x(\tau)$ to another position $x(\tau + T) = 0$ in the least possible time $T = T^\circ$. We shall denote by G the region $\zeta > 0$, $T^\circ[x, \zeta] < \infty$ in the $\{x, \zeta\}$-space.

We assume that at the initial instant $\tau = t_0$, the pursued player has at his disposal a safety margin $\nu(t_0) - \varepsilon(t_0)$ differing from $\nu(t_0)$ by a small quantity $\varepsilon(t_0) = \varepsilon^\circ > 0$. Since the point $\{y^\circ - z^\circ, \mu^\circ - \nu^\circ\}$ belongs to the region G, so does $\{y^\circ - z^\circ, \mu^\circ - (\nu^\circ - \varepsilon^\circ)\}$. The function $T^\circ[x, \zeta]$ will be continuous in x and ζ in the vicinity of any point in G, hence

$$0 \leqslant T^\circ[y^\circ - z^\circ, \mu^\circ - \nu^\circ] - T^\circ[y^\circ - z^\circ, \mu^\circ - (\nu^\circ - \varepsilon^\circ)] \leqslant \delta_1(\varepsilon^\circ) \tag{2.3}$$

$$\lim_{\varepsilon^\circ \to 0} \delta_1(\varepsilon^\circ) = 0$$

Let us choose, at the initial instant $\tau = t_0$, a sufficiently small ε° which will then define $T^\circ[x^\circ, \zeta^\circ]$ where

$$x = y - z, \qquad \zeta = \mu - (\nu - \varepsilon) \tag{2.4}$$

and let us also select $\vartheta_\varepsilon(t_0) = t_0 + T^\circ[x^\circ, \zeta^\circ]$. We shall choose at any instant $\tau \geq t_0$ such $\varepsilon(\tau)$, that

$$\vartheta_\varepsilon(\tau) = \tau + T^\circ[x(\tau), \zeta(\tau)] = \vartheta_\varepsilon(t_0) = \text{const} \tag{2.5}$$

The pursued player will be aware of all $y(\tau)$, $z(\tau)$, $\mu(\tau)$ and $\nu(\tau)$ which came into being up to the instant τ, therefore in accordance with the properties of $T^\circ[x, \zeta]$, (2.5) may yield a unique $\varepsilon(\tau) = \varepsilon[y(\tau), z(\tau), \mu(\tau), \nu(\tau)]$, satisfying it. We shall consider this magnitude $\varepsilon(\tau)$ used in constructing the strategy $v = v_\varepsilon$, as a new variable.

When investigating our problem of interception and constructing the required strategy $v = v_\varepsilon$, we shall find that an important part is played by the controls $u = u^\circ$ and $v = v^\circ$ aiming the motions of (1.1) and (1.2) at the point of contact of two regions of accessibility, $H^{(1)}[y(\tau), \mu(\tau), \vartheta_\varepsilon]$ and $H^{(2)}[z(\tau), \nu(\tau) - \varepsilon(\tau), \vartheta_\varepsilon]$; these controls are given by [2]

$$u^\circ(\tau) = \frac{w_\tau^\circ(\tau)\mu(\tau)}{\mu(\tau) - (\nu(\tau) - \varepsilon(\tau))}, \quad v^\circ(\tau) = \frac{w_\tau^\circ(\tau)(\nu(\tau) - \varepsilon(\tau))}{\mu(\tau) - (\nu(\tau) - \varepsilon(\tau))} \tag{2.6}$$

where $w_\tau^\circ(t)$, $t \geqslant \tau$ is a solution of the problem (2.1), (2.2) when

$$x(\tau) = y(\tau) - z(\tau), \quad \zeta(\tau) = \mu(\tau) - (\nu(\tau) - \varepsilon(\tau))$$

We shall now determine the strategy $v = v_\varepsilon$. Let

$$\eta(\tau) = \frac{\varepsilon(\tau)}{\nu(\tau)}, \qquad \xi(\tau) = \frac{\zeta(\tau)}{\nu(\tau)} \tag{2.7}$$

$$\varphi(\eta, \xi) = \frac{(1 + \eta/\xi)}{(1 - \eta)^2} \qquad (\xi > 0, \ \eta < 0) \tag{2.8}$$

$$v_\varepsilon(\tau) = \begin{cases} \varphi(\eta(\tau), \xi(\tau))\, v^\circ(\tau) & (t_0 \leqslant \tau \leqslant t^*) \\ 0 & (\tau > t^*) \end{cases} \tag{2.9}$$

where t^* denotes the first instant of time when

$$\frac{\nu(\tau) - \varepsilon(\tau)}{\zeta(\tau)} = \gamma$$

The magnitude γ will be defined in Section 3. This completes the formal part of constructing the strategy $v = v_\varepsilon$, and we shall show in Section 3 that this strategy does indeed solve the stated problem.

Below we give basic operations which shall be utilized in Section 3 in investigating the constructed strategy. First we shall obtain, utilizing the conditions (2.5),

$$\frac{d\zeta}{d\tau} = -\frac{1}{2\zeta}(\|w^\circ\|^2 + 2(w^\circ, \delta w)) \qquad (\delta w = w - w^\circ) \tag{2.10}$$

Next we shall find $d\varepsilon/d\tau$. From (2.4) we have

$$\frac{d\varepsilon}{d\tau} = \frac{d\zeta}{d\tau} - \frac{d\mu}{d\tau} + \frac{d\nu}{d\tau} \tag{2.11}$$

Let us transform its right-hand side using the relations

$$\frac{d\mu}{d\tau} = -\frac{\|u(\tau)\|^2}{2\mu}, \qquad \frac{d\nu}{d\tau} = -\frac{\|v(\tau)\|^2}{2\nu} \tag{2.12}$$

following from (1.3). Relations (2.10) to (2.12) yield

$$\frac{d\varepsilon}{d\tau} = \frac{d\zeta}{d\tau} - \frac{d\mu}{d\tau} + \frac{d\nu}{d\tau} = -\frac{1}{2\zeta}(\|w^\circ\|^2 + 2(w^\circ, \delta w)) + \frac{\|u\|^2}{2\mu} - \frac{\|v\|^2}{2\nu} \tag{2.13}$$

let

$$\delta u = u - u^\circ, \quad \delta v = v - v^\circ \tag{2.14}$$

Since $u - v = w$ and $u^\circ - v^\circ = w^\circ$, we have $\delta w = \delta u - \delta v$. Taking these into account we can transform (2.13) into

$$\frac{d\varepsilon}{d\tau} = \frac{\|\delta u\|^2}{2\mu} - \frac{\|\delta v\|^2}{2(\nu - \varepsilon)} + \frac{\varepsilon\|v\|^2}{2(\nu - \varepsilon)\nu} \tag{2.15}$$

while (2.7) yields

$$\frac{d\eta}{d\tau} = \frac{1}{\nu}\left(\frac{d\varepsilon}{d\tau} - \eta\frac{d\nu}{d\tau}\right), \quad \frac{d\xi}{d\tau} = \frac{1}{\nu}\left(\frac{d\zeta}{d\tau} - \xi\frac{d\nu}{d\tau}\right) \tag{2.16}$$

where $d\varepsilon/d\tau$, $d\nu/d\tau$ and $d\zeta/d\tau$ are defined from (2.15),(2.12) and (2.10), respectively.

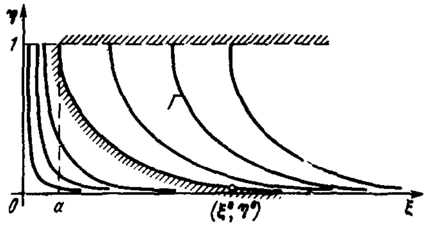

Fig. 1

Consider a function

$$V(\eta, \xi) = \eta - \eta(\xi) \qquad (2.17)$$

where $\eta(\xi)$ is one of the family of curves defined by

$$\frac{d\eta}{d\xi} = -\frac{\eta(2-\eta+1/\xi)}{(1-\eta)^2} \qquad (0 \leqslant \eta < 1,\ \xi > 0) \qquad (2.18)$$

which are shown in Fig. 1 . We note that

$$\psi(\eta, \xi) = \frac{\eta(2-\eta+1/\xi)}{(1-\eta)^2} \geqslant 0 \qquad (0 \leqslant \eta < 1,\ \xi > 0) \qquad (2.19)$$

$$\varphi(\eta, \xi) \equiv \psi(\eta, \xi) + 1 \qquad (2.20)$$

Let us find $dV/d\tau$ near a curve belonging to the above family assuming that, by (2.9), $v = \varphi v^\circ$. We have

$$\frac{dV}{d\tau} = \frac{d\eta}{d\tau} - \frac{d\eta}{d\xi}\frac{d\xi}{d\tau} = \frac{d\eta}{d\tau} + \psi(\eta, \xi)\frac{d\xi}{d\tau}$$

Taking into account (2.16), we obtain

$$\frac{dV}{d\tau} = \frac{1}{\nu}\left(\frac{d\varepsilon}{d\tau} - \frac{d\nu}{d\tau}\eta\right) + \frac{\psi}{\nu}\left(\frac{d\zeta}{d\tau} - \frac{d\nu}{d\tau}\xi\right) = \frac{1}{\nu}A - \frac{1}{\nu}\frac{d\nu}{d\tau}\eta \qquad (2.21)$$

$$\left(A = \frac{d\varepsilon}{d\tau} + \psi\frac{d\zeta}{d\tau} - \psi\xi\frac{d\nu}{d\tau}\right)$$

Relations (2.15),(2.12) and (2.10) yield

$$A = \frac{\|\delta u\|^2}{2\mu} - \frac{\|\delta v\|^2}{2(\nu-\varepsilon)} + \frac{\|v\|^2\varepsilon}{2\nu(\nu-\varepsilon)} + \psi\left[-\frac{1}{2\zeta}(\|w^\circ\|^2 + 2(w^\circ, \delta w)) + \xi\frac{\|v\|^2}{2\nu}\right] = \frac{\|\delta u\|^2}{2\mu} - \frac{\psi(w^\circ, \delta u)}{\zeta} - \frac{\|\delta v\|^2}{2(\nu-\varepsilon)} + \frac{\varepsilon\|v\|^2}{2\nu(\nu-\varepsilon)} + \frac{\psi}{\zeta}(w^\circ, \delta v) - \frac{\psi}{2\zeta}\|w^\circ\|^2 + \frac{\psi}{2\nu}\xi\|v\|^2$$

This shows that $A \geq 0$, therefore we have, taking into account (2.11) and the fact that $d\nu/d\tau \leq 0$,

$$\left(\frac{dV}{d\tau}\right)_{v=\varphi v^\circ} \geqslant 0 \qquad (2.22)$$

3. Analysis of the strategy $v = v_\varepsilon$. We shall show that the strategy constructed above, solves the stated problem. Before proving it, we shall note the following fact. In constructing the strategy $v = v_\varepsilon$, the pursued player assumes that he has, at any instant τ , a safety margin $\nu(\tau) - \varepsilon(\tau)$ differing from the actual safety margin by $\varepsilon(\tau)$, that this assumed safety margin never exceeds the value of the actual one and that, consequently, the inequality

$$\varepsilon(\tau) \geqslant 0 \qquad (t_0 \leqslant \tau \leqslant \vartheta) \qquad (3.1)$$

must hold at any instant up to the time of interception.

This inequality will be verified later during the analysis of the strategy v_ε .

Let δ be a positive number specified in advance. We shall show that there exists ε° and γ (see (2.9)) such that the strategy $v = v_{\varepsilon}$ guarantees that interception will occur not sooner than in the time $T_{u^{\circ}v^{\circ}}(t_0) - \delta$: We shall first determine ε°. Let $\delta_1 = \delta/2$, then from (2.3) it follows that such $\varepsilon^{\circ}(\delta_1) > 0$ can be found, that

$$T^{\circ}[y^{\circ} - z^{\circ}, \mu^{\circ} - \nu^{\circ}] - T^{\circ}[y^{\circ} - z^{\circ}, \mu^{\circ} - (\nu^{\circ} - \varepsilon^{\circ})] = T_{u^{\circ}v^{\circ}}(t_0) - T^{\circ}[y^{\circ} - z^{\circ}, \mu^{\circ} - (\nu^{\circ} - \varepsilon^{\circ})] \leqslant \delta_1 = \delta / 2 \tag{3.2}$$

We shall now assume that $\varepsilon^{\circ} > 0$ has been chosen in accordance with (3.2) . In this case, ν°, ε° and ζ° which define the point $\{\eta^{\circ} = \varepsilon^{\circ}/\nu^{\circ}, \xi^{\circ} = \zeta^{\circ}/\nu^{\circ}\}$ and a curve belonging to the family (2.18) passing through this point, will all be known at the instant $\tau = t_0$. We can also assume without loss of generality, that in the beginning the control $v = v_{\varepsilon}$ is chosen according to the first Formula of (2.9). Then the inequality $(dV/d\tau_{v=\varphi v^{\circ}}) \geqslant 0$ will imply that the point $\{\eta(\tau), \xi(\tau)\}$ can only move upward from one curve of (2.18) to the next one and, for at least as long, as

$$\frac{\nu(\tau) - \varepsilon(\tau)}{\zeta(\tau)} = \frac{1 - \eta(\tau)}{\xi(\tau)} \geqslant \gamma$$

holds. Consequently the control is chosen in the form $v = \varphi v^{\circ}$.

At the same time the point $\{\eta(\tau), \xi(\tau)\}$ will remain, at all times, within the region Γ (see Fig. 1). Indeed, the point $\{\eta, \xi\}$ cannot appear below the curve containing $\{\eta^{\circ}, \xi^{\circ}\}$, since $(dV/d\tau)_{v=\varphi v^{\circ}} \geqslant 0$, and it cannot intersect the straight line $\eta = 1$, since in this case we would have, remembering that $\xi(\tau) \geq a > 0$, $(1 - \eta(\tau))/\xi(\tau) = 0 < \gamma$.

Let $\gamma > 0$ be a constant. We shall prove the following assertion: (A). It, for any τ $(t_0 \leqslant \tau \leqslant \tau^* < \vartheta_{\varepsilon})$ the inequality

$$\frac{1 - \eta(\tau)}{\xi(\tau)} \equiv \frac{\nu(\tau) - \varepsilon(\tau)}{\zeta(\tau)} \geqslant \gamma \tag{3.3}$$

holds and $v = \varphi v^{\circ}$, then no interception takes place within the interval $[t_0, \tau^*]$.

We shall use two additional propositions during the proof of (A).

1°. Let

$$0 < M \leqslant T^{\circ}[x(\tau), \zeta(\tau)] \leqslant N \qquad (M, N = \text{const}) \tag{3.4}$$

$$\zeta(\tau) \geqslant \alpha > 0, \qquad t_0 \leqslant \tau \leqslant \tau^* \tag{3.5}$$

Then, for any τ belonging to $[t_0, \tau^*]$ we have $x(\tau) \neq 0$.

This follows from the properties of the function $T^{\circ}[x, \zeta]$.

2°. If at $t_0 \leqslant \tau \leqslant \tau^* < \vartheta_{\varepsilon}$ the inequality

$$\eta(\tau) \leqslant 1 - \beta \qquad (\beta = \text{const} > 0) \tag{3.6}$$

holds, $v = \omega v^{\circ}$ and the point $\{\eta(\tau), \xi(\tau)\}$ remains within Γ, then

$$\nu(\tau) \geqslant \alpha > 0, \qquad t_0 \leqslant \tau \leqslant \tau^* < \vartheta_{\varepsilon} \tag{3.7}$$

To prove (3.7), we shall consider Expression

$$\frac{d\nu}{d\tau} = -\frac{\|v\|^2}{2\nu} = -\frac{\varphi^2 \|v^{\circ}\|^2}{2\nu} = -\varphi^2 (1 - \eta)(\nu - \varepsilon)\frac{\|w^{\circ}\|^2}{2\zeta^2} \tag{3.8}$$

Since $\{\eta, \xi\} \in \Gamma$, we have $\xi \geqslant a > 0$ (see Fig. 1). Taking now (3.6) into account, we obtain

$$\varphi(\eta(\tau), \xi(\tau)) = \frac{1 + \eta(\tau)/\xi(\tau)}{(1 - \eta(\tau))^2} \leqslant K_1 \qquad (K_1 = \text{const}) \tag{3.9}$$

We shall now show that

$$\frac{\| w^\circ(\tau) \|^2}{\zeta^2(\tau)} \leqslant K_2 \qquad (K_2 = \text{const}, \ t_0 \leqslant \tau^* \leqslant \tau^* < \vartheta_\varepsilon) \tag{3.10}$$

Indeed, $\|w^\circ(\tau)\|^2$ and $\zeta^2(\tau)$ are [2] positive definite quadratic forms in vector coordinates $x(\tau)$ with variables continuous in τ and coefficients, bounded when $t_0 \leqslant \tau \leqslant \tau^* < \vartheta_\varepsilon$, hence the ratio of these forms satisfies (3.10). Relations (3.8) to (3.10) now imply that $d\nu / d\tau \geqslant -K_3\nu$, $t_0 \leqslant \tau \leqslant \tau^* < \vartheta_\varepsilon$ and the condition $\nu^\circ > 0$ yields (3.7).

We shall now prove the assertion (A). We have shown that in the case under consideration, point $\{\eta(\tau), \xi(\tau)\}$ remains within Γ, hence $\eta(\tau) = \varepsilon(\tau)/\nu(t) \geq 0$ and $\varepsilon(\tau) \geq 0$, and the condition (3.1) holds. We have, for any point within Γ, $\xi(\tau) \geq a > 0$, hence (3.3) yields $1 - \eta \geq \gamma a$, $\eta \leq 1 - \gamma a$. 2° now implies that $\nu(\tau) \geq \alpha > 0$ ($t_0 \leq \leq \tau \leq \tau^*$) and, as $\xi(\tau) = \zeta(\tau)/\nu(\tau) \geq a$, we have $\zeta(\tau) \geq \alpha a > 0$, i.e. we have (3.5). We note that at any $\tau_* < \vartheta_\varepsilon$ the condition (3.4) follows from (2.5), therefore 1° yields $x(\tau) = y(\tau) - z(\tau) \neq 0$ ($t_0 \leqslant \tau \leqslant \tau^* < \vartheta_\varepsilon$), which completes the proof (A).

Thus, if inequality (3.3) is valid for any $\tau < \vartheta_\varepsilon$, then interception would occur not sooner than at the instant $\vartheta_\varepsilon(t_0) = T_{u^\circ v^\circ}(t_0) - \delta/2 + t_0$.

However, let Eq.

$$\frac{[\nu(\tau^*) - \varepsilon(\tau^*)]}{\zeta(\tau^*)} = \gamma$$

be satisfied for the first time at the instant τ^*.

We need only consider the case where

$$t_0 \leqslant \tau^* \leqslant \vartheta_\varepsilon - {}^1/_2\,\delta \tag{3.11}$$

since the strategy $v = v_\varepsilon$ constructed must guarantee that interception will occur not sooner than at the instant $T_{u^\circ v^\circ}(t_0) + t_0 - \delta = \vartheta_\varepsilon(t_0) - {}^1/_2\,\delta$. But by (2.9), for $\tau \geq \tau^*$ we have $v(\tau) \equiv 0$, so that the minimum time which elapses from the instant τ^* to interception is $T^\circ[x, (\tau^*), \mu(\tau^*)]$. Moreover, from (3.3) and (2.4) we have $\mu(\tau^*) = \zeta(\tau^*) + \gamma\zeta(\tau^*)$. Thus, at the instant τ^* we have

$$\tau^* + T^\circ[x(\tau^*), \zeta(\tau^*)] = \vartheta_\varepsilon(t_0), \qquad \tau^* + T^\circ[x(\tau^*), \zeta(\tau^*) + \gamma\zeta(\tau^*)] = \vartheta^\circ \tag{3.12}$$

Here ϑ° is the earliest possible instant of interception (under the condition $v(\tau) = 0$, $\tau > \tau^*$).

We shall now prove another assertion.

3°. Let

$$T[x, \zeta] = K = \text{const} > 0, \tag{3.13}$$

hold for $\{x, \zeta\} \in G$. Then

$$T^\circ[x, \zeta] - T^\circ[x, \zeta + \gamma\zeta] = \omega(x, \zeta, \gamma) \to 0 \qquad \text{for } \gamma \to 0 \tag{3.14}$$

is uniform in all x and ζ satisfying (3.13).

With x and ζ fixed, condition (3.14) holds by virtue of the continuity of $T^\circ[x, \zeta]$ in ζ, therefore it only remains to show that $\omega(x, \zeta, \gamma) \to 0$ as $\gamma \to 0$ uniformly in x and ζ. Let us write an equation defining $T = T^\circ[x, \zeta]$

$$(D_T x, x)^{1/2} - \zeta = 0 \tag{3.15}$$

Here $(D_T x, x)$ is a positive definite from whose coefficients are continuous functions of T ($T > 0$). Let us write (3.15) as

$$(D_T q, q)^{1/2} - 1 = 0, \qquad (q = x / \zeta) \tag{3.16}$$

Since $(D_K q, q)$ is a positive definite quadratic form, it follows that the set Q of all

q for which $(D_K q, q)^{\frac{1}{2}} - 1 = 0$, is closed. Eq.

$$(D_T q, q)^{1/2} - (1 + \gamma) = 0$$

gives $T = T^\circ [x, \zeta + \gamma\zeta]$

The root of this equation depends continuously on q and γ, hence $\omega(x, \zeta, \gamma)$ is also a continuous function of q and γ only, defined for at least $q \in Q$ and $\gamma \geq 0$.

Since q is a member of a closed set Q, then (3.14) holds uniformly for all q belonging to Q or, in other words, is uniform in all x and ζ satisfying (3.13), and this completes the proof 3°.

From (2.5) and (3.11) it follows that the case under consideration $1/2\, \delta \leqslant K \leqslant \vartheta_\varepsilon (t_0)$. Using 3° we can find, for any K on the segment under consideration, such $\gamma(K)$ that $\omega(x, \zeta, \gamma) \leq \frac{1}{2}\delta$ is uniform in x and ζ.

Now let us take $\gamma = \min_K \gamma\ (K)(1/2\, \delta \leqslant K \leqslant \vartheta_\varepsilon (t_0))$. We can show that for this γ interception occurs not sooner than in the time $T_{u^\circ v^\circ} (t_0) - \delta$. In fact $\omega(x, \zeta, \gamma) \leqslant 1/2\, \delta$ i.e. $T^\circ [x, \zeta] - T^\circ [x, \zeta + \gamma\zeta] \leqslant 1/2\, \delta$, so that (see (3.12))

$$\vartheta_\varepsilon(t_0) - \vartheta^\circ \leqslant \delta / 2, \qquad \vartheta \geqslant \vartheta_\varepsilon - \delta / 2 = t_0 + T_{u^\circ v^\circ} (t_0) - \delta$$

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